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NONSTATIONARY CRITICAL LAYER AND NONLINEAR INSTABILITY IN A PLANAR POISEUILLE FLOW

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One of the promising directions in the nonlinear instability theory of shear flows is related to the study of critical layers (CL) [1-6]. Stationary waves with a viscous nonlinear CL have been studied in most detail [2, 3]. Analysis of nonstationary processes of practical interest was carried out for significant simplifying restrictions [4-6]. Thus, the nonlinear development of a wave in a channel and in a boundary layer was treated only in the limiting case of a strongly nonlinear CL near a stationary one [5]. To solve the problem of generation of turbulence in these flows, however, it is necessary to have some idea of the evolution of an initially linear wave. To study nonlinear instability in a planar Poiseuille flow we use below an approach similar to that of [6] for weather instability. We consider the development of long waves, represented on the (R, α) plane by points in the neighborhood of the upper branch of the neutral curve of the linear theory (α, wave number; and R, Reynolds number). For these waves it is possible to consider independently CL and viscous regions near the channel walls. Based on analyzing a nonstationary CL, we obtain equations describing the time evolution of a wave. The transition is traced from a linear viscous CL to a wave strongly nonlinear in the increasing amplitude. As is well known, stability problems with hydrodynamic flows are largely similar in that wave—particle interactions are generated in the plasma [7-9]. In the present paper the plasma-hydrodynamic analogy provides the wave energy in a Poiseuille flow, making it possible to interpret the results obtained from the point of view of general wave theory.

1. Starting Relations. We write down the equations for a viscous incompressible fluid in the form [10]

$$\partial\zeta/\partial t + u\partial\zeta/\partial x + v\partial\zeta/\partial y = \nu\Delta\zeta; \tag{1.1}$$

$$\Delta\Psi = -\zeta, \tag{1.2}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$; ζ, is the flow vorticity, Ψ is the stream function, introduced by the relations $u = \partial\Psi/\partial y$, $v = -\partial\Psi/\partial x$; and $\nu = 1/R \ll 1$ is the reciprocal Reynolds number (all the variables are assumed to have been reduced to dimensionless form). Putting

$$\Psi = \int U(y)\,dy + \psi,$$

where $U(y) > 0$ is the velocity profile in a stationary Poiseuille flow between the walls $y = 0$ and $y = 2$, we obtain the following equation for ψ:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\Delta\psi - U''\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x} + \nu\Delta^2\psi \tag{1.3}$$

(the prime denotes differentiation with respect to y). Considering a wave periodic in x, we denote the complex amplitude of the Fourier harmonic by a variable with subscript n (n = 1, 2...): $\psi_n(y, t) = \langle\psi\exp(-in\alpha\xi)\rangle$, etc., where $\xi = x - ct$, c is the phase velocity of the wave, and $\langle\ldots\rangle$ is the average over a period. In the linear approximation the profile $\psi_1(y)$ of a

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neutral sinusoidal wave in an ideal fluid satisfies the Rayleigh equation [10]. The vorticity profile in this wave can be represented in the form

$$\zeta_1 = -[U''/(U-c)]\psi_1. \tag{1.4}$$

At resonance $y = y_c$ ($U(y_c) = c$) the function ψ_1 is bounded and, consequently, ζ_1 endures a jump. The scales of the narrow critical layers formed near $y = y_c$ with account of wave nonstationarity, viscosity, or nonlinearity can be represented, respectively, in the form [2-6]

$$d_t = \gamma_c/\alpha U_c', \quad d_l = (\nu/\alpha U_c')^{1/3}, \quad d_n = (B/U_c')^{1/2},$$

where $U_c' = U'(y_c)$; $\gamma_c = |\psi_c^{-1} d\psi_c/dt|$; $\psi_c = \psi_1(y_c, t)$; and $B = 2|\psi_c|$ is the fluctuation amplitude of the stream function at $y = y_c$. We further consider flows with an isolated CL, whose scale $d_c = \max(d_t, d_l, d_n)$ is small in comparison with the distance to the channel walls. In this case the weak wave nonstationarity, viscosity, and nonlinearity can be taken into account within perturbation theory everywhere except in the CL region and the viscous regions near the channel walls.

2. Nonstationary Critical Layer. Consider a CL in a wave with a slowly varying amplitude ($\gamma_c \ll \alpha c$). Following [2, 3], we introduce the small parameter ε determining the order of magnitude of the oscillation amplitude of ψ. To simplify the analysis we assume that the process is characterized by the scales d_t and d_l, which coincide in order of magnitude with the CL nonlinearity scale $d_n \sim \varepsilon^{1/2}$. Correspondingly, we introduce the normalized viscosity $\bar{\nu} = \nu/\varepsilon^{3/2} \sim 1$ and the slow time $\tau = \varepsilon^{1/2}t$. An external solution of (1.3) is constructed in the form of a series in powers of the small parameter $\varepsilon^{1/2}$:

$$\psi = \varepsilon \sum_{n=1}^{\infty} \left(A_{n\pm}^{(0)}\varphi_{an} + B_{n\pm}^{(0)}\varphi_{bn}\right) e^{in\alpha\xi} + \text{к. с.} + \varepsilon^{3/2}\psi^{(1)} + \ldots, \tag{2.1}$$

where $\varphi_{an} = \varphi_a(n\alpha, c; y)$, $\varphi_{bn} = \varphi_b(n\alpha, c; y)$ are Tollmien functions [3, 6], $A_{n\pm}^{(0)}(\tau)$ and $B_{n\pm}^{(0)}(\tau)$ are complex coefficients, the subscripts ($\pm$) refer to values in the regions $y > y_c$ and $y < y_c$, respectively, and c.c. denotes the complex conjugate expression. Substituting (2.1) into (1.3), we obtain for the amplitudes $\psi_n^{(l)}$ equations of the form

$$\frac{\partial^2\psi_n^{(l)}}{\partial y^2} - \left(n^2\alpha^2 + \frac{U''}{U-c}\right)\psi_n^{(l)} = F_n^{(l)}, \tag{2.2}$$

where $F_n^{(l)}$ are expressed in terms of the series (2.1), whose number is less than l; in particular,

$$F_n^{(1)} = -\frac{U''}{in\alpha(U-c)^2}\left(\frac{dA_{n\pm}^{(0)}}{d\tau}\varphi_{an} + \frac{dB_{n\pm}^{(0)}}{d\tau}\varphi_{bn}\right). \tag{2.3}$$

The solution of (2.2) is written in the form

$$\psi_n^{(l)} = A_{n\pm}^{(l)}\varphi_{an} + B_{n\pm}^{(l)}\varphi_{bn} + \varphi_{an}\int^y \varphi_{bn}F_n^{(l)}dy - \varphi_{bn}\int^y \varphi_{an}F_n^{(l)}dy, \tag{2.4}$$

where $A_n^{(l)}$ and $B_n^{(l)}$ are complex coefficients. Consider the behavior of the solution (2.1) near resonance points. Keeping in mind the structure of the Tollmien function for $\eta = y - y_c \to 0$, it is seen that the integrals in (2.4) can be represented as series expansions in powers of η and $\ln|\eta|$. It is agreed to choose the integration constants so that these expansions contain no constants. We then obtain for $\psi_n^{(1)}$

$$\psi_n^{(1)} = A_{n\pm}^{(1)}\varphi_{a1} + B_{n\pm}^{(1)}\varphi_{b1} + \frac{1}{i\alpha n}\frac{dB_{n\pm}^{(0)}}{d\tau}(\ln|\eta| + 1) + O(\eta\ln|\eta|). \tag{2.5}$$

Following the common scheme of matched asymptotic expansions, we transform in the external solution for $\eta \to 0$ to the CL variable $Y = \eta/\varepsilon^{1/2}$. As a result we obtain for the harmonic amplitudes of the stream function the following representation for $Y \to \pm\infty$:

$$\Psi_n = \varepsilon B_{n\pm}^{(0)} + \varepsilon^{3/2}\ln\varepsilon^{1/2}\left[\frac{U_c''}{U_c'}B_{n\pm}^{(0)}Y + \frac{1}{i\alpha n}\frac{U_c''}{U_c'^2}\frac{dB_{n\pm}^{(0)}}{d\tau}\right] + \varepsilon^{3/2}\left[\frac{U_c''}{U_c'}B_{n\pm}^{(0)}\times\right. \tag{2.6}$$

$$\left.\times Y\ln|Y| + A_{n\pm}^{(0)}Y + B_{n\pm}^{(1)} + \frac{1}{i\alpha n}\frac{U_c''}{U_c'}\frac{dB_{n\pm}^{(0)}}{d\tau}(\ln|Y| + 1) + O(1/Y)\right] + \ldots,$$

where $U_c'' = U''(y_c)$. To find the quantity $O(1/Y)$ appearing with $\varepsilon^{3/2}$ in (2.6) it is necessary to consider $\psi^{(l)}$ for $l \geqslant 2$. The outer expansion of the mean of the total stream function acquires the form

$$\langle\Psi\rangle = \varepsilon^{1/2}cY + \varepsilon\frac{1}{2}U_c'Y^2 + \varepsilon^{3/2}\left[\frac{1}{6}U_c''Y^3 + O(1/Y)\right] + \dots \tag{2.7}$$

In the CL region we transform to the reference system of the neutral wave and put $\zeta = -U_c' + \varepsilon^{1/2}\Omega$. The solution is sought in the form

$$\Psi = \varepsilon^{1/2}cY + \varepsilon\Psi^{(1)} + \varepsilon^{3/2}\ln\varepsilon^{1/2}\Psi^{(2)} + \varepsilon^{3/2}\Psi^{(3)} + \dots \tag{2.8}$$

Determining $\Psi^{(1)}$, $\Psi^{(2)}$ and matching expansions (2.7), (2.8), and (2.6) to orders ε and $\varepsilon^{3/2}\ln\varepsilon^{1/2}$ for $Y \to \pm\infty$, we obtain

$$B_{n+}^{(0)} = B_{n-}^{(0)} \equiv B_n^{(0)},$$

$$\Psi^{(1)} = \frac{1}{2}U_c'Y^2 + \sum_{n=1}^{\infty} 2\,\mathrm{Re}\,(B_n^{(0)}e^{in\alpha\xi}), \quad \Psi^{(2)} = \frac{U_c''}{U_c'}B_n^{(0)}Y + \frac{1}{i\alpha n}\frac{U_c''}{U_c'^2}\frac{dB_n^{(0)}}{d\tau}; \tag{2.9a}$$

$$\partial^2\Psi^{(3)}/\partial Y^2 = -\Omega. \tag{2.9b}$$

The following equation follows from (1.1) for the vorticity after time $t \sim 1/\varepsilon^{1/2}$

$$\frac{\partial\Omega}{\partial\tau} = \bar{\nu}\frac{\partial^2\Omega}{\partial Y^2} - U_c'Y\frac{\partial\Omega}{\partial\xi} + 2\sum_{n=1}^{\infty}\mathrm{Re}\left[in\alpha B_n^{(0)}e^{in\alpha\xi}\right]\frac{\partial\Omega}{\partial Y}. \tag{2.10}$$

The evolution of vorticity occurs in the velocity field, determined by the stream function (2.9a). The longitudinal component of the velocity field is given by the primary flow near resonance points, while the transverse one is determined by the external inviscid solution for $y \to y_c$. Using (2.6), the jumps $A_{n+}^{(0)} - A_{n-}^{(0)}$, $B_{n+}^{(1)} - B_{n-}^{(1)}$ can be expressed in terms of the values of $\Psi_n^{(3)}$ and $\partial\Psi_n^{(3)}/\partial Y$ for $Y \to \pm\infty$. Taking then into account (2.9b), we obtain

$$A_{n+}^{(0)} - A_{n-}^{(0)} = -\lim_{T\to\infty}\int_{-T}^{T}\Omega_n dY; \tag{2.11a}$$

$$B_{n+}^{(1)} - B_{n-}^{(1)} = \lim_{T\to\infty}\int_{-T}^{T}\left(Y\Omega_n + \frac{U_c''}{U_c'}B_n^{(0)}\right)dY. \tag{2.11b}$$

We next multiply (2.10) by $\exp(-im\alpha\xi)$, average the equation obtained over a wave period, and integrate over Y from $-\infty$ to $+\infty$. Using (2.6), (2.7), (2.11b) is transformed to the form

$$B_{n+}^{(1)} - B_{n-}^{(1)} = -\frac{1}{in\alpha U_c'}\frac{\partial}{\partial\tau}\int_{-\infty}^{\infty}\Omega_n dY. \tag{2.12}$$

Unlike the outer expansions for the stationary problem, constructed in [3], Eqs. (2.6), (2.7) contain derivatives of the amplitude with respect to time and include the presence of multiple harmonics. We stress that (2.1) does not contain terms $\sim\varepsilon^{1/2}$, which in the stationary theory describe the deformation of the primary flow, generating a jump in the mean vorticity upon transition through the CL (see relations (2.3), (2.4) in [3]). To explain the variation of the mean vorticity in a nonstationary CL, we take into account that a solution of (2.10), which for $Y \to \pm\infty$ transforms to a flow with $\langle\Omega\rangle = -U_c''Y + H_\pm(\tau)$, can be constructed in the form

$$\Omega \to -U_c''Y + H_\pm(\tau) + \sum_{l=1}^{\infty}\frac{K_l(\xi,\tau)}{Y^l}, \tag{2.13}$$

where K_l is a periodic function of ξ. Substituting (2.13) into (2.10) and equating coefficients with identical powers of Y, we obtain for $H_\pm$ the equations $dH_\pm/d\tau = 0$ and find K_l explicitly. We assume that initially the wave amplitude is quite small. For $\tau = 0$, then, the mean flow coincides with the primary one and $H_\pm(\tau) \equiv 0$. The absence of a limiting transition to a stationary CL is explained by the fact that for $dB_n^{(0)}/d\tau \to 0$ the edge of the step on the profile $\langle\Omega\rangle + U_c''Y$ is "misplaced" for bounded Y.

According to (2.10), even in case of a sinusoidal wave ($B_n^{(0)} = 0$ for $n = 2, 3, \dots$) the harmonic amplitudes of the vorticity $|\Omega_n|$ in a nonlinear CL are quantities of the same order of magnitude. In this case it follows from (2.11a) that to match the outer and inner expansions in the principal part of (2.1) one must, in the general case, include harmonics with $n \geqslant 2$.

In the theory of hydrodynamic instability it is common to reduce the circulation feature in the Rayleigh equation to a logarithmic phase jump occurring in φ_b. Using (2.1), (2.11a), we obtain for harmonics with n = 1

$$\Phi = \frac{U_c'}{U_c''} \operatorname{Im}\left[\frac{1}{B_1^{(0)}} \int_{-\infty}^{\infty} \Omega_1 dY\right], \tag{2.14}$$

where Φ is the logarithmic phase decrement upon passing from $y_c - 0$ to $y_c + 0$.

3. Long-Wave Perturbations. Account of a Viscous Sublayer. We consider a wave with a symmetric profile ψ_n and construct a solution on the channel halfwidth $0 < y < 1$. The boundary-value problem is significantly simplified when the resonance point is near the boundary ($y_c \ll 1$). We show that in this case the wave is almost sinusoidal in the outer regions. We neglect fluid adhesion to the channel walls. It is natural to assume that Eq. (2.10) and the matching conditions (2.11a), (2.12) are satisfied if the CL is isolated from the boundary ($d_c/y_c \ll 1$). Using explicit forms of φ_a and φ_b, one can verify that for $U_c' \sim U_c'' \sim 1$ the coefficients $A_1^{(0)}$ and $B_1^{(0)}$ differ strongly in magnitude: $|B_1^{(0)}/A_{1-}^{(0)}| \sim y_c \ll 1$. In the central CL zone we put, according to (1.4), $|\zeta_1| \sim \varepsilon |B_1^{(0)}|/d_c$. The following estimate is obtained in this case

$$\int_{-\infty}^{\infty} \Omega_1 dY \sim B_1^{(0)}. \tag{3.1}$$

As a result we reach the relation

$$|A_{1+} - A_{1-}| \ll |A_{1\pm}|, \ |B_{1+} - B_{1-}| \ll |B_{1\pm}|, \tag{3.2}$$

where $A_{1\pm} = \varepsilon A_{1\pm}^{(0)}$; $B_{1\pm} = \varepsilon B_1^{(0)} + \varepsilon^{3/2} B_{1\pm}^{(1)}$. From the physical point of view, inequalities (3.2) correspond to a small contribution of resonances with a vortex wave to the velocity fluctuations in the outer regions. In this case the CL role as a source of multiple harmonics is clear. The multiple harmonics are small when they are induced flow waves (c is not an eigenvalue of the boundary-value problem with wave number $n\alpha$). Indeed, putting in the induced solution $|A_{n\pm}| \sim |A_{n+} - A_{n-}|$, we obtain an estimate of the contribution of multiple harmonics to (2.1): $|A_{n\pm}| \sim |B_{1\pm}| \ll |A_{1\pm}|$, $|B_{n\pm}| \sim y_c |A_{n\pm}| \ll |B_{1\pm}|$ for $n \geqslant 2$.

The condition of nearness of resonance points to the channel wall in a planar Poiseuille flow is satisfied for long waves ($\alpha^2 \ll 1$). We first find the neutral linear waves in an ideal flow, for which the circulation feature usually has the form $A_{1+} = A_{1-}$, $B_{1+} = B_{1-}$ (this rule is equivalent to the circulation feature in the Rayleigh equation in the principal value sense). The boundary conditions are written in the form $\psi_1(0) = \psi_1'(1) = 0$. Using the matched expansion method, one can find the phase velocity of the wave c and the amplitude profile of the stream function f(y):

$$c = \frac{\alpha^2}{U_0'} \int_0^1 U^2 dy; \tag{3.3a}$$

$$f(y) = \begin{cases} k_a \varphi_a(\alpha, c; y) + k_b \varphi_b(\alpha, c; y), & 0 < y < 1, \\ U(y) + O(\alpha^2), & \alpha^2 \ll y \leqslant 1, \end{cases} \tag{3.3b}$$

where $k_a \approx U_0'$; $k_b \approx c$; $U_0' = U'(0) \approx U_c' > 0$. The condition of nearness of resonance points to the boundary and the restriction on the CL scale acquire the form

$$y_c = c/U_0' \sim \alpha^2 \ll 1, \quad U_0' d_c / c \ll 1. \tag{3.4}$$

Another special region is generated in a viscous flow near the walls, the viscous sublayer (VS) [1]. Far from the boundary the neutral wave can be primarily described by the Rayleigh equation if one uses the boundary condition [5]

$$\psi_1 = -(i+1)\varkappa \psi_1'|_{y=0}, \tag{3.5}$$

where $\varkappa = (\nu/2\alpha c)^{1/2}$ is the scale of a linear VS. A weak nonstationarity of waves leads to a small correction to (3.5). The treatment of CL provided above remains valid if the CL and VS regions are isolated:

$$\varkappa/y_c \ll 1. \tag{3.6}$$

4. Wave Interaction with a Critical Layer and with a Viscous Sublayer. Taking into account the weak effect of CL and VS on the wave structure in the outer regions, the nonstationary solution (1.3) is represented for $|\eta| \gg d_c$ in the form

$$\psi = (1/2)af(y)e^{i\alpha\xi} + \text{complex conjugate} + w, \tag{4.1}$$

where w is a small correction and $a(t)$ is the slowly varying complex wave amplitude

$$|a^{-1}da/dt| = \gamma_c \ll \alpha c. \tag{4.2}$$

Substituting (4.1) into (1.3) and neglecting nonlinear and viscous terms, we obtain for $w_1 = \langle w \exp(-i\alpha\xi)\rangle$ the equation

$$\frac{\partial^2 w_1}{\partial y^2} - \left(\alpha^2 + \frac{U''}{U-c}\right) w_1 = -\frac{1}{2}\frac{da}{dt}\frac{U''f}{i\alpha(U-c)^2}. \tag{4.3}$$

The solution of (4.3) is represented in the form $w_1 = A_w^{\pm}\varphi_a + B_w^{\pm}\varphi_b + \tilde{w}$, where $|A_w^{\pm}| \ll k_a|a|$, $|B_w^{\pm}| \ll k_b|a|$, $\tilde{w}$, and w is a particular solution of (4.3). The function $\tilde{w}$ is constructed in the same manner as the singular part of the solution of (2.4). Then $A_{1\pm} = (1/2)ak_a + A_w^{\pm}$, $B_{1\pm} = (1/2)\alpha k_b + B_w^{\pm}$, and the circulation rule in (4.3) acquires the form

$$A_w^+ - A_w^- = -\varepsilon\int_{-\infty}^{\infty}\Omega_1 dY,\quad B_w^+ - B_w^- = -\frac{\varepsilon^{3/2}}{i\alpha U_c'}\frac{\partial}{\partial\tau}\int_{-\infty}^{\infty}\Omega_1 dY. \tag{4.4}$$

From the symmetry of the profile $\psi_1(y)$ and from (3.5) follow the boundary conditions

$$w' = 0|_{y=1},\ w = -(1/2)(i+1)\varkappa f'a|_{y=0}. \tag{4.5}$$

The resulting homogeneous boundary-value problem (4.3)-(4.5) has nontrivial solutions of the form (3.3). To derive orthogonality relations we multiply (4.3) by f(y) and integrate over the region $0 < y < y_c - \delta$, $y_c + \delta < y < 1$, where δ is an infinitely small positive quantity. Integrating by parts, taking into account (4.4), and using for the transformation of the contribution of w the constancy of the Wronskian $\varphi_a'\varphi_b - \varphi_b'\varphi_a = 1$, we obtain

$$\frac{U_0'}{2i\alpha}\frac{da}{dt} = -k_b(A_w^+ - A_w^-) + k_a(B_w^+ - B_w^-) - \frac{1}{2}a(i+1)\varkappa f'^2(0). \tag{4.6}$$

In deriving (4.6) we took into account the relation

$$\lim_{\delta\to 0}\left\{\int_0^1 \frac{U''f^2}{(U-c)^2}dy - \frac{2k_b^2U_c''}{U_c'^2\delta}\right\} \approx -U_0',$$

where the region $y_c - \delta < y < y_c + \delta$ was excluded from the integral. Conditions (4.2), (3.1) make it possible to neglect the contribution of $B_w^+ - B_w^-$ in (4.6). As a result, we obtain the following equation for the complex wave amplitude

$$\frac{da}{dt} = (1-i)\alpha\varkappa U_0' a + \frac{2i\alpha c}{U_0'}\int_{-\infty}^{\infty}\varepsilon\Omega_1 dY. \tag{4.7}$$

From Eqs. (4.7), (2.10) one can eliminate the constant frequency correction $\Delta\omega = \alpha\varkappa U_0'$, generated by the viscous wave interaction with the channel walls. For this it is sufficient to transform to new variables

$$a_N = a\exp(i\Delta\omega t),\quad \xi_N = \xi - (\Delta\omega/\alpha)t,\quad \eta_N = \eta - \Delta\omega/\alpha U_c'.$$

Transforming in (2.10), (4.7) to nonnormalized variables, and omitting the subscript N, we finally obtain

$$dA/dt = (\gamma_V + (1/\pi)\gamma_L\Phi)A; \tag{4.8}$$

$$\frac{\partial\tilde{\zeta}}{\partial t} = \nu\frac{\partial^2\tilde{\zeta}}{\partial\eta^2} - U_c'\eta\frac{\partial\tilde{\zeta}}{\partial\xi} - \alpha B\sin\alpha\xi\frac{\partial\tilde{\zeta}}{\partial\eta}, \tag{4.9}$$

where A = $|a|$ is the wave amplitude, $\gamma_L = -\pi \frac{\alpha c U''_c}{U'^2_c}$; $\gamma_V = U'_0(\nu\alpha/2c)^{1/2}$; the fluctuating amplitude of the stream function in the CL is B = cA, $\tilde{\zeta}$ is the deviation of the vorticity from its value in the primary flow at level U = c + ($\Delta\omega/\alpha$), and

$$\Phi = -\frac{2U'_c}{U''_c B} \int_{-\infty}^{\infty} \langle \tilde{\zeta} \sin \alpha \xi \rangle \, d\eta$$

is the logarithmic phase jump (2.14), expressed in terms of the variables $\tilde{\zeta}$ and A. In (4.8) we use the solution (4.9) with asymptotics at $\eta/d_c \to \pm\infty$ similar to (2.13). In deriving (4.8), (4.9) we took into account the symmetry relation $\tilde{\zeta}(\xi, \eta, t) = -\zeta(-\xi, -\eta, t)$, due to which the wave phase is constant (without loss of generality, we took arg $a_N = 0$).

According to (4.8) the wave increment $\gamma = A^{-1}dA/dt$ consists of two parts. The first of them (γ_V) is determined by the viscous wave interaction with the channel walls and is positive, and the second is proportional to the logarithmic phase jump, inducing a vorticity in the CL region. The applicability conditions of the equations obtained reduces to inequalities (3.4), (3.6), and (4.2). As follows from (3.6), for fixed $\alpha^2 \ll 1$ the viscosity must be quite small ($\nu \ll 2\alpha c^3/U_0'^2 \sim \alpha^7$). The latter is in agreement with the well-known conclusion of the linear theory that near the asymptotes of the lower branches of the neutral curve ($\nu \sim \alpha^7$) the CL and VS form one viscous region [11].

5. Wave-Particle Interaction. Energy Relations. For $\nu = 0$ the system (4.8), (4.9) practically coincides with the equations describing nonlinear Landau damping of electrostatic waves in a plasma [12-15]. In this case the vorticity profile in a CL plays the role of a plasma velocity distribution function of resonance particles.

The plasma-hydrodynamic analogy makes it possible to construct energy relations for long waves in a channel in the spirit of the general wave theory. According to [16] the energy of a plasma wave is determined without taking into account resonance particles. We correspondingly call the energy of hydrodynamic wave the flow energy increment in the external (with respect to CL and VS) regions, generated by exciting a given wave. Since $\bar{u} = \langle u \rangle - U \ll U$, one can write the normalized energy density in the form

$$H = \int_0^1 \left(\frac{1}{2} \langle u_\sim^2 \rangle + \frac{1}{2} \langle v_\sim^2 \rangle + U\bar{u} \right) dy,$$

where $u_\sim = u - \langle u \rangle$; $v_\sim = v - \langle v \rangle \equiv v$ and from the integral we excluded the CL and VS regions, whose boundaries are, respectively, $y = y_c \pm \delta_c(x, t)$ and $y = \delta_*(x, t)$, the material lines, with $d_c \ll \delta_c \ll y_c$, $\varkappa \ll \delta_* \ll y_c$. We find an explicit form of H indirectly, by means of conservation laws and Eq. (4.7). Integrating the energy balance equation in an ideal flow [17] over the outer region, we obtain

$$\frac{\partial H}{\partial t} = c \langle u_\sim v_\sim \rangle \Big|_{y_c-\delta_c}^{y_c+\delta_c} + c \langle u_\sim v_\sim \rangle_{y=\delta_*}. \tag{5.1}$$

In deriving (5.1) we neglected the variation in U, v_1, and the profile of fluctuating pressure across the CL, as well as the wave nonstationarity near the channel walls. As follows from (5.1), the incoming power in the wave is determined by the jumps in Reynolds stresses at the singular regions. The jump in u_1 in crossing through the CL is found from (1.2), and the Reynolds stress at the VS boundary can be calculated by using (3.5); we then have

$$\langle u_\sim v_\sim \rangle \Big|_{y_c-\delta_c}^{y_c+\delta_c} = -\int_{-\delta_c}^{\delta_c} 2 \operatorname{Re}(\zeta_1 v_1^*) \, d\eta; \tag{5.2}$$

$$\langle u_\sim v_\sim \rangle_{y=\delta_*} = -\frac{1}{2} \varkappa \alpha U_0'^2 |a|^2. \tag{5.3}$$

Comparison of (5.1)-(5.3) and the expression for $d|a|^2/dt$, following from (4.7), makes it possible to write the wave energy in the form

$$H = -\frac{1}{4} c U_0' A^2.$$

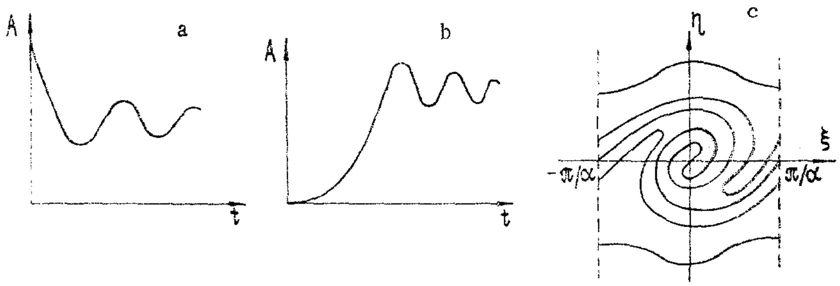

Fig. 1.

The wave energy seems to be negative.* It was noted in [18] that waves in an ideal Poiseuille flow with a piecewise linear velocity profile possess a negative energy. The conclusion obtained by us justifies considering the flow model with a piecewise-linear velocity profile as an idealization of a real flow, corresponding to the transparent medium approximation [19] in terms of wave theory. The latter, however, is valid only for waves near the upper branch of the neutral curve.

The energy determination given above makes it possible to interpret the wave behavior upon interaction with special regions. According to (5.1), (5.3) the wave enhancement upon interaction with a VS is a wave instability with negative energy in a system with positive dissipation [16]. To explain the CL role consider linear waves in an ideal fluid ($\nu = 0$). We put $\tilde{\zeta} = -U_c''\eta + \zeta_\sim$ in (4.9), and we linearize this equation in the oscillation amplitude. Solving then (4.8), (4.9) by the Laplace transform method (similarly to [6]), it can be shown that for $t \to \infty$ the logarithmic phase jump is $\Phi \to -\pi$, and $A \sim \exp(-\gamma_L t)$. Thus, γ_L in (4.8) is the Landau damping decrement.

The wave is damped for $U_c'' < 0$ and is amplified for $U_c'' > 0$. As noted in [9], for $U_c'' < 0$ near resonance points there occurs in the unperturbed flow an inversion of the velocity distribution of fluid particles, leading to negative dissipation of the wave energy; a wave with negative energy must damp ($\gamma_L > 0$) under these conditions. When $U_c'' > 0$ the dissipation is positive and an instability develops.

To analyze nonlinear damping (amplification) of a wave in an inviscid flow, we also use results of the theory of plasma waves. For purposes of comparison with plasma problems, it must be taken into account that for $\nu = 0$ Eq. (4.9) expresses the vortex conservation law by fluid particles. The particle trajectories are found from the equations

$$d\xi/dt = U_c'\eta, \quad d\eta/dt = \alpha c A \sin \alpha\xi.$$

The current lines on the (ξ, η) plane form a "cat's eye." For $A = \text{const}$ the particles perform rotations near the "pupil" with frequency $\omega_{tr} = \alpha(cU_c'A)^{1/2}$. When the vorticity perturbations in the CL region are initially small ($\tilde{\zeta}(0, \xi, \eta) \approx -U_c''\eta$), according to the conclusions of plasma theory [14] the strong wave amplitude ($\omega_{tr}(t = 0) \gg \gamma_L$) decays for $U_c'' < 0$, as shown on Fig. 1a. The stabilization of the weak wave instability ($\omega_{tr}(t = 0) \ll \gamma_L$) for $U_c'' > 0$ is illustrated in Fig. 1b. An estimate for the maximum amplitude can be obtained from the condition $\gamma_L \sim \omega_{tr}$. Material lines, having at $t = 0$ the form of straight lines $u = C$, transform after confining the instability to strongly twisted spirals (Fig. 1c),† being also lines of vorticity levels ($\tilde{\zeta} = -U_c''C$).

6. Neutral Curve of the Linear Theory. The study of instability in the ideal fluid approximation is of interest for explaining the possible behavior of the system. In a real Poiseuille flow, however, viscosity plays an essential role. We show that Eqs. (4.8), (4.9) lead to the well-known expression for the asymptote of the upper branch of the neutral curve.

For $d_\eta \ll d_l$ stationary waves in the reference system with profile $\tilde{\zeta}$ will be sought in the form of an expansion in the wave amplitude:

$$\tilde{\zeta} = -U_c''\eta + \zeta^{(1)} + \zeta^{(2)} + \zeta^{(3)} + \cdots \tag{6.1}$$

In the first approximation we obtain from (4.9) the well-known expression in the linear viscous CL theory for the amplitude of the vorticity profile:

*Waves with negative energy were first considered in hydrodynamics in [20].
†A similar pattern was obtained in [4] for a shear layer and in [6] for weather waves.

$$\frac{\partial^2\zeta_1^{(1)}}{\partial Y_0^2} - iY_0\zeta_1^{(1)} = \frac{1}{2}\, i\,\frac{U_c''B}{U_c'd_l}, \tag{6.2}$$

where $Y_0 = \eta/d_l$. After transforming to the Fourier representation in the variable Y_0, Eq. (6.2) acquires the form

$$\frac{dG}{dq} + q^2G = -\frac{1}{2}\, i\,\frac{U_c''B}{U_c'd_l}\,\delta(q)\left[G = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\zeta_1^{(1)}e^{iqY_0}\,dY_0\right], \tag{6.3}$$

where $\delta(q)$ is the delta-function. The solution (6.3) is written in the form

$$G = -\frac{1}{2}\, i\,\frac{U_c''B}{U_c'd_l}\int\limits_{-\infty}^{q}\delta(q')\exp[q'^3/3 - q^3/3]\,dq'.$$

Taking into account further the relation

$$\int\limits_{-\infty}^{\infty}\zeta_1^{(1)}d\eta = 2\pi d_l G(0),$$

we obtain, as in the inviscid problem, $\Phi = -\pi$, and for $U_c'' < 0$ we have the Lin equation for the asymptote of the upper branch of the neutral curve:

$$\nu = \frac{2\pi^2\alpha c^5 U_c''^2}{U_c'^6} \sim \alpha^{11}.$$

The phase velocity of the wave (3.3a) also coincides with that found by Lin.

7. Quasistationary Approximation. Development of Instability in a Viscous Flow. For a constant wave amplitude diffusion-dissipation processes tend to stabilize the $\tilde{\zeta}$ distribution in the CL region. Assuming stability of the stationary solutions of (4.9) and under conditions of their quite quick establishment in (4.9), one can neglect $\tilde{\zeta}_t$ and use results of studying stationary CL [2, 3] in solving the nonstationary problem.

Putting in (4.9) $\partial/\partial t = 0$ and transforming to the variables

$$X = \alpha x, \quad Y_* = \eta/d_n, \quad \bar{\Omega} = \tilde{\zeta}/U_c''d_n, \tag{7.1}$$

we obtain an equation for the nonlinear stationary CL, investigated in [3]:

$$Y_*\frac{\partial\bar{\Omega}}{\partial X} + \sin X\frac{\partial\bar{\Omega}}{\partial Y_*} = \lambda_c\frac{\partial^2\bar{\Omega}}{\partial Y_*^2},$$

where $\lambda_c = (d_l/d_n)^3$. After transforming to normalized time and amplitude

$$\tau_* = (|\gamma_L|/\pi)\,t, \quad s = \frac{1}{\lambda_c^{2/3}} = \left(\frac{\alpha}{\nu}\right)^{2/3}\frac{c}{U_c'^{1/3}}A$$

Eq. (4.8) transforms to the form†

$$\frac{ds}{d\tau_*} = s[\beta - \Phi(s)\operatorname{sgn} U_c''], \tag{7.2}$$

where $\Phi = -2\int\limits_{-\infty}^{\infty}\langle\bar{\Omega}\sin X\rangle\, dY_*$ is the logarithmic phase jump, written in the variables of (7.1),‡ and $\beta = \pi\gamma_V/|\gamma_L|$. To find Φ for $s \ll 1$ one can use the perturbation procedure described in Sec. 6. Calculating $\zeta^{(2)}$ and $\zeta^{(3)}$, we obtain

$$\Phi = -\pi + ks^2\left[k = \frac{\pi}{6}\left(\frac{3}{2}\right)^{1/3}\Gamma(1/3) \approx 1.6\right],$$

†We note that in [5] Zhigulev studies waves with a strongly nonlinear CL, whose growth is related with account of nonstationarity of the CL in the wave reference system. In the present approach solutions of this type can be obtained by neglecting the term $ds/d\tau_*$ in (7.2) and taking into account the change in Φ due to the contribution of $\tilde{\zeta}_t$ to (4.9). Unlike [5], the method of analysis adopted in the present work makes it possible to consider the formation of a nonlinear CL in the wave.

‡ Using (2.9b), one can show that Φ coincides with that introduced in [3].

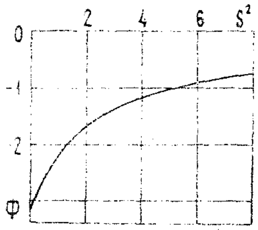

Fig. 2.

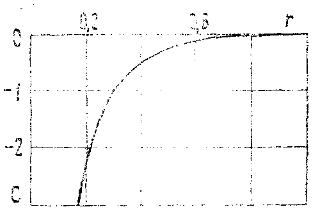

Fig. 3.

where $\Gamma(1/3)$ is the value of the Gamma-function. According to [3], $\Phi \to -4.2\ s^{-3/2}$ for $s \gg 1$. Figure 2 shows the curve $\Phi(s^2)$, constructed from the data of [3]. It can be shown that for arbitrary s values the function Φ is well approximated by the simple equation

$$\Phi = -\pi/(1 + (4k/3\pi)s^2)^{3/4}.$$

The description of a nonlinear CL within condition (7.2) does not contradict the isolation condition of CL and VS regions in a wide range of amplitude variation, since for waves near the top branch of the neutral curve ($\beta \sim 1$) this condition acquires the form $d_n/y_c \sim \alpha^{4/3}s^{1/2} \ll 1$.

Consider the development of instability in a stationary viscous Poiseuille flow $U = 1 - (1 - y)^2$. In the region above the asymptote of the upper branch of the neutral curve, Eq. (7.2) has two equilibrium states in the linear theory ($0 < \beta < \pi$). One of them is stable and corresponds to an unperturbed flow, and the other describes stationary waves of finite amplitude, as constructed in [3]. These waves are unstable to small amplitude perturbations. When $\beta > \pi$, the unperturbed flow is unstable within the linear approximation. An increment at the initial stage of instability increases at any β with amplitude and tends to a constant value for $s \to \infty$, which is explained by the weakened stabilization action of the CL. If the infinitely small perturbations increase ($\beta > \pi$) and the instability is initially linear ($ks^2(0) \ll \beta - \pi$), one obtains the following expression for the transition time to amplitudes $s \gg 1$

$$\tau_* \approx \frac{1}{\pi}\left[-\frac{1}{r}\ln s(0) + \frac{1}{r+1}\ln s + C(r)\right], \tag{7.3}$$

where $r = (\beta/\pi) - 1$ is the supercriticality of the wave, and the function $C(r)$ is shown in Fig. 3. In the case of a "purely explosive" instability ($r = 0$) one obtains the following expression for τ_* at $s(0) \ll 1$ and $s \gg 1$:

$$\tau_* \approx (1/\pi)\,[0.98/s^2(0) - 1.16\ln s(0) + \ln s + 0.17]. \tag{7.4}$$

Equation (7.2) is, strictly speaking, valid only for $d_n \ll y_c$. However, the behavior of the solutions found shows that the amplitude growth must lead to a merging of the CL central zone with the VS region if the increment β is to remain constant. The merging takes place at $A \sim c/U_0'$ during a time which can be estimated from Eqs. (7.3), (7.4).

We discuss briefly the applicability limits of the quasistationary approximation for the CL region. We assume that $\nu \sim \alpha^{11}$, the CL is nonlinear ($s \gtrsim 1$), and, correspondingly, $\gamma \sim \gamma_V \sim \alpha^5$. Denoting by k_l ($l = 1 - 4$) the terms in Eq. (4.9) ordered from left to right, we obtain the estimate

$$k_1/k_{3,4} \sim \alpha^{2/3}s^{-1/2},\ k_1/k_2 \sim \alpha^{2/3}s. \tag{7.5}$$

For $s \sim 1$ the contribution of $\tilde{\zeta}_t$ to (4.9) is asymptotically small. In a strongly nonlinear CL ($s \gg 1$) small scales in comparison with d_n are generated on the vorticity profile [2], and the estimate (7.5) needs to be refined.

We discuss now nonlinear effects in the VS, leading to breakdown of (3.5) and a change in γ_V. Assuming the profile deformation of the mean flow to be uniform along x, we obtain for $\bar{u}$ the equation

$$\frac{\partial \bar{u}}{\partial t} - \nu\frac{\partial^2 \bar{u}}{\partial y^2} = -\frac{\partial}{\partial y}\langle u_\sim v_\sim\rangle. \tag{7.6}$$

The condition $t_d\gamma \sim \alpha^{-2} \gg 1$ ($t_d = (\Delta y)^2/\nu$ is the diffusion time for scale Δy) is satisfied in the region separating the CL and VS, of transverse size α^2. Neglecting then in (7.7) the diffusion terms and solving (4.3), one can find $\bar{u}$ in explicit form

$$\overline{u} = \frac{1}{4} \frac{f^2 U''}{(U-c)^2} A^2. \tag{7.7}$$

This expression is obtained for $\overline{u}$ in the region confined between the CL and the mid-region of the channel. Near the walls the characteristic scale is $\varkappa \sim \alpha^4$ and, consequently, $t_d\gamma \sim \alpha^2 \ll 1$. In this case the operator in the left-hand side of (7.6) can be assumed to be the diffusion one, and, putting according to (43.5) $\psi_1 \sim \varkappa A$, one obtains the estimate $\overline{u} \sim A^2/\alpha^2$. The transition to the value $\overline{u} \sim A^2$, determined by (7.7) in the outer region, occurs on the interval $\Delta y \sim \alpha^3 \gg \varkappa$, where the viscosity and nonstationarity are equally important in (7.6). The derivation of the boundary condition (3.5) is based on using in the VS the simplified equation for the amplitude profile

$$d^4\psi_1/dy^4 + (i\alpha c/\nu)d^2\psi_1/dy^2 = 0. \tag{7.8}$$

It can be shown that the contribution of nonlinear terms with $\overline{u} \neq 0$ to (7.8) cannot be assumed small for $A \sim \alpha^2$, i.e., when the CL coincides with the VS.

Similarly to the way the profile of $\overline{u}$ was treated above near the walls, one can proceed to develop the profile of the mean vorticity in the CL. The inequality $\gamma t_d \sim \alpha^{2/3} \ll 1$ is satisfied in the kernel of the nonlinear CL, and the process is quasistationary. In this case the step of [3] is generated on the mean vorticity profile. In the outer regions $\gamma t_d \gg 1$ and the development of $\langle\tilde{\zeta}\rangle$ is determined by the "inviscid" equations. For the size of the "avalanche" region of the step edges Δy the condition $t_d \sim \gamma$ results in the estimate $\Delta y \sim \alpha^3$. Since $\Delta y/d_n \gg 1$, Φ can be calculated from the stationary theory.

The conclusions obtained in the present work for wave evolution in time can be applied to analyze spatial growth if the replacement $t \to x/v_g$ is made (v_g is the wave group velocity). For large Reynolds numbers the stationary profile in a Poiseuille flow usually does not get established, and the results obtained are valid under the assumption of quasiparallel flow during the nonlinear stage of instability evolution.

The concept of a weak wave interaction with a critical layer and with a viscous sublayer can be used to treat instability in a boundary layer. From the point of view of general wave theory in nonequilibrium media it is interesting that the energy of a hydrodynamic wave, determined from the position of the plasma-hydrodynamic analogy, is negative, while the wave behavior is found to be in agreement with the sign of its energy.

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STABILITY OF STATIONARY SPATIALLY PERIODIC CONVECTIVE MOTIONS IN A PLANE VERTICAL LAYER

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UDC 532.516:536.25

A wide range of studies has been dedicated to the stability of plane-parallel convective motions in viscous liquid layers (see [1, 2]). It is known that in those cases where instability is of a monotonic character, it leads to development of stationary spatially periodic motions. Clever et al. [3, 4] studied stability of finite amplitude secondary motions. In [5-8] the stability of convective swell was considered, while [9] treated hexagonal cells which develop in horizontal layers due to an equilibrium crisis. In these studies stability was determined by solution of the spectral problem obtained by applying the Halerkin method to the linearized problem for perturbations. The present study will examine the stability of stationary spatially periodic motions in a planar vertical layer in the presence of lateral heating. The increments of the least stable perturbation will be determined from the time asymptote of the solution of the linearized perturbation problem, which will be constructed by the grid method [10, 11]. Calculations are performed for Prandtl number by the grid method [10, 11]. Calculations are performed for Prandtl number Pr = 1 over the Grashof number range 500 < Gr < 2000. The dependence of the increment on quasiwave number is obtained, the boundaries of the stability region are defined for spatially periodic secondary motions, and the main types of perturbations producing instability are determined.

1. We will consider an infinite vertical layer filled by a viscous incompressible fluid. On the solid boundaries of the layer ($y = \pm d$) constant but different temperatures $T = \pm\Theta$ are maintained (the x axis is directed vertically upward, and the y axis is horizontal). In dimensionless form we write the system of equations for two-dimensional convection:

$$\partial\Phi/\partial t = \Delta\Phi + \mathrm{Gr}\,\partial T/\partial y + D(\Phi, \Psi)/D(x, y); \tag{1.1}$$

$$\Delta\Psi = -\Phi; \tag{1.2}$$

$$\partial T/\partial t = (1/\mathrm{Pr})\Delta T + D(T, \Psi)/D(x, y), \tag{1.3}$$

where $D(f, g)/D(x, y) = (\partial f/\partial x)\partial g/\partial y - (\partial f/\partial y)\partial g/\partial x$; Ψ, flow function; Φ, vorticity. The similarity parameters are the Grashof number Gr and Prandtl number Pr. Assuming the flow to be closed (no pumping of liquid along the layer) the boundary conditions have the form

$$\Psi = \partial\Psi/\partial y = 0,\ T = \pm 1 \ \text{at}\ \ y = \pm 1. \tag{1.4}$$

We also require that all functions remain finite at infinity

$$|\Phi|,\ |\Psi|,\ |T| < \infty \ \text{ as }\ x \to \pm\infty. \tag{1.5}$$

Boundary problem (1.1)-(1.5) always has a solution

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